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LETTER TO THE EDITOR

Finite-size scaling theory for non-linear relaxation

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Abstract. We present a finite-size scaling theory for the non-linear relaxation of a simple relaxational model.

An interesting aspect of critical phenomena is the effect of finite size on the bulk thermodynamic and dynamical properties of a system. These effects are of intrinsic interest since experiments are always performed on finite systems and have observable consequences for relatively thin fields (Lutz *et al* 1978). One striking feature involves the 'rounding off' of an infinity in the susceptibility or specific heat at the critical point of the bulk system to a finite value which depends on the size of the system. A successful phenomenological finite-size scaling theory has been developed for thermodynamics by Fisher (1971) and has recently been derived by Suzuki (1977) by renormalisation group arguments. Suzuki (1977) also extended this finite-size scaling to dynamics for the linear relaxation time of kinetic Ising models. It should be noted that these finite-size scaling theories are not only relevant for predicting the critical behaviour of finite systems, but have also been used to obtain estimates for the *bulk* critical exponents. For example, Nightingale and Blote (1980) recently obtained quite good estimates for the critical exponents for the $d=2$ q-state Potts model, while Yalabik and Gunton (1979) obtained a reasonable estimate of the dynamical exponent z for the $d=2$ kinetic Ising model. Thus a study of finite-size effects is proving quite useful for a variety of reasons.

In this note we extend Suzuki's theory to obtain a finite-size scaling theory for the *non-linear* relaxation (Suzuki 1971, Binder 1973) rate for kinetic Ising models. Although studies of non-linear effects are much more difficult than the corresponding linear problems, some progress has already been made for bulk systems, including both a phenomenological scaling theory (Racz 1976, Fisher and Racz 1976) and recent renormalisation group calculations (Bausch and Janssen 1976, Kawasaki 1976, Suzuki 1976, Bausch *et al* 1979, Yamada *et al* 1979). We would hope that a finite-size scaling theory for the non-linear relaxation could provide some additional insight into these phenomena. One aspect of particular interest in what follows will be the asymptotic behaviour of the non-linear relaxation time in a finite system.

We begin by recalling that for infinite systems the renormalisation group calculations lead to a scaled equation of motion for the order parameter m, from which the scaling form for the relaxation time τ can be derived. This scaling takes the form

$$\tau=\epsilon^{-\nu z}\Phi_{\infty}(x) \tag{1}$$

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with

$$x = m(0)/\epsilon^{\beta} \tag{2}$$

where we use the standard notation for critical exponents and where ϵ is the reduced temperature and $m(0)$ is the initial value of the order parameter. The renormalisation group result (1) is in agreement with earlier predictions based on scaling arguments (Racz 1976, Fisher and Racz 1976, Kretschmer *et al* 1976). It should be stressed, however, that the actual linear and non-linear exponents do not follow just from scaling arguments. Their values depend on the asymptotic properties of the function Φ_∞ which itself requires the solution of the renormalised dynamical problem. It has generally been assumed (Racz 1976, Fisher and Racz 1976, Bausch and Janssen 1976, Kawasaki 1976, Suzuki 1976) that in the linear regime ($x \ll 1$)

$$\Phi_\infty(x) \sim c \tag{3}$$

where c is a constant, and in the non-linear regime ($x \gg 1$)

$$\Phi_\infty(x) \sim x^{-1}. \tag{4}$$

These assumptions imply respectively that

$$\tau^{l} \sim \epsilon^{-\nu z}; \qquad \Delta_l = \nu z \tag{5}$$

$$\tau^{nl} \sim \epsilon^{\beta-\nu z}; \qquad \Delta_{nl} = \nu z - \beta \tag{6}$$

with a crossover between these two limiting behaviours occurring for $x \sim 1$.

Until recently (Bausch *et al* 1979) a valid justification of expression (4) had not been given. Indeed, the arguments presented by Fisher and Racz (1976) are from an *a priori* point of view rather doubtful, since their starting point is a renormalised instantaneous equation of motion of the Ginzburg–Landau type in which non-Markovian terms are neglected. Renormalisation group calculations show, however, that non-Markovian terms should also be included (Bausch and Janssen 1976, Yamada *et al* 1979). Nevertheless the ϵ expansion (Bausch *et al* 1979) seems to give results consistent with equation (6).

Turning now to finite systems, a scaling theory for the linear relaxation has been derived by Suzuki (1977)†. His predictions are in reasonable agreement with recent calculations (Nightingale and Blote 1980, Yalabik *et al* 1979). For a d-dimensional lattice of volume L^d, the linear relaxation time is predicted to have a maximum value

$$\tau^{l} \sim L^{z} \tag{7}$$

instead of its bulk divergence $\tau^{l} \sim \xi^{z}$, where ξ is the correlation length. This linear relaxation time crosses over to the bulk behaviour (5) for

$$y \equiv \epsilon L^{1/\nu} \sim 1. \tag{8}$$

The natural question then arises as to how the finite size will smooth out the divergence of the non-linear relaxation time and correspondingly, what will be its asymptotic behaviour. A simple argument provides a sensible answer to this question. We have in principle two characteristic times

$$t_1 \sim \epsilon^{-\nu z} \tag{9}$$

† It should be noted that Suzuki's derivation does not give the general scaling form of Fisher, since it yields for the exponent λ which characterises the shift in the critical point $\lambda = 1/\nu$, whereas in general this is not always the case.

for which a crossover from non-linear to linear behaviour occurs in the bulk limit, and

$$t_2 \sim L^z \tag{10}$$

for which the crossover from the bulk region to the finite-size region occurs. In the finite-size limit $y \ll 1$, we have $t_1 \gg t_2$ and only t_2 is left as a characteristic time in which the change from linear to non-linear behaviour can occur. If we assume a smooth matching at $t = t_2$ of a power law behaviour of the time-dependent order parameter

$$m(t) \sim m(0) t^{-\beta/\nu z} \tag{11}$$

in the non-linear region ($t < t_2$), with an exponential decay in the linear region ($t > t_2$), we obtain equally important contributions to the non-linear relaxation time τ^{nl} from both regions. For example, an estimate of the contribution to τ^{nl}

$$\tau^{nl} \equiv \int_0^\infty \mathrm{d}t \frac{m(t)}{m(0)} \tag{12}$$

in the linear region is

$$\tau^{nl} \sim \int_{t_2}^\infty \mathrm{d}t \frac{m(t)}{m(0)} = \frac{m(t_2)}{m(0)} \int_{t_2}^\infty \frac{m(t)}{m(t_2)} \mathrm{d}t = L^{-\beta/\nu} \tau^l = L^{z-\beta/\nu}. \tag{13}$$

We might also note that to obtain the scaling relation (6) for infinite systems it has been argued that the main contribution to τ^{nl} comes from the long-time relaxation in the linear region. In that case, however, there is a difficulty in that the crossover time t_1 goes to infinity when $T \to T_c$ (Fisher and Racz 1976). Since it is this crossover time going to infinity which gives rise to the difference between the linear and non-linear exponents, this heuristic argument is somewhat dubious for the bulk system. For finite systems however, even for the extreme limit for the non-linear relaxation in which $\epsilon = 0$, t_2 remains finite, so the argument seems reasonable.

We now turn to a formal analysis of the general scaling form for τ and discuss different limiting behaviour of the scaling function which unifies the results (5)–(7) and (12) and allows us to discuss the different crossover phenomena. This generalises Suzuki's (1977) finite-size dynamical scaling theory for the linear relaxation time. From his results we have the scaling form

$$m(t) = \epsilon^\beta f(h\epsilon^{-\beta\delta}, \epsilon L^{1/\nu}, t\epsilon^{\nu z}) \tag{14}$$

where h is the initial magnetic field. We use the equilibrium equation of state for the infinite system

$$h = m(0)^\delta g\left(\frac{m(0)}{\epsilon^\beta}\right) \tag{15}$$

to eliminate h in terms of the initial condition $m(0)$ in (14). This yields

$$m(t) = \epsilon^\beta F_1(x, y, t\epsilon^{\nu z}) \tag{16}$$

where x and y are defined by equations (2) and (8). Therefore we obtain

$$\tau = \int_0^\infty \mathrm{d}t \frac{m(t)}{m(0)} = \epsilon^{-\nu z} \Phi_1(x, y) \tag{17}$$

where

$$\Phi_1(x, y) = \int_0^\infty \mathrm{d}t' \frac{F_1(x, y, t')}{x}; \qquad t' = t\epsilon^{\nu z}. \tag{18}$$

Equation (17) is the scaled form of τ which can alternatively be written as

$$\tau = L^z \Phi_2(x, y) \tag{19}$$

where

$$\Phi_2(x, y) = y^{-\nu z} \Phi_1(x, y). \tag{20}$$

In the bulk limit

$$\Phi_1(x, y) \xrightarrow[y \gg 1]{} \Phi_\infty(x) \tag{21}$$

so that we recover equation (1) and as a consequence equations (5) and (6).

In the finite-size limit we assume for the linear regime

$$\Phi_2(x, y) \xrightarrow[\substack{y \ll 1 \\ x \ll 1}]{} \text{constant} \tag{22}$$

so that equation (7) is derived. In the non-linear regime

$$\Phi_2(x, y) \xrightarrow[\substack{y \ll 1 \\ x \gg 1}]{} x^{-1} y^{-\beta} \tag{23}$$

from which equation (13) is recovered. Alternatively we have

$$\Phi_1(x, y) \xrightarrow[\substack{y \ll 1 \\ x \ll 1}]{} y^{\nu z} \tag{24}$$

and

$$\Phi_2(x, y) \xrightarrow[\substack{y \ll 1 \\ x \gg 1}]{} x^{-1} y^{\nu z - \beta}. \tag{25}$$

For $y \sim 1$, equations (7) and (13) cross over to equations (5) and (6).

Finally we note that the result (13) might be used to calculate the non-linear exponent Δ_{nl} by a Monte Carlo simulation of a finite-size system, similar to the calculation of Δ_l by Yalabik and Gunton (1979).

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